Design of CFAR Radars using Compressive Sensing

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***Abstract*—In this work we propose the GLRT-MP algorithm which combines compressed sensing techniques and classical detection theory and explores its application to sparse arrays. Sparse arrays are large undersample arrays with nonuniform spacing that provides high resolution at the cost of high sidelobes. Compressed sensing techniques are able to minimize the undesired effects of the large array, while classical detection theory provides a way to perform detection while maintaining a desired false alarm probability. We provide analysis of the GLRT when the noise power is known and unknown, the latter which will allow one to design a CFAR radar. We provide numerical results to verify our results.**

I. Introduction

It is well known that a uniform linear array (ULA) with half wavelength spacing ensures the absence of grating lobes in the visible region of the array pattern. ULAs also produce low sidelobes, a desirable trait during the beamforming stages of target detection. In contrast, sparse arrays with large, nonuniform inter-element spacing produces large sidelobes but achieve higher resolution due to the large aperture [16]. However, due to the large sidelobes, the beamformer often experience false peaks, which increases the probability of false alarm.

In recent years, compressive sensing (CS) techniques tailored for sparse localization frameworks [4], [8] were shown to be able to cope with the spatial undersampling of sparse arrays [5]. This exciting result enables the radar designer to implement large undersampled arrays without needlessly increasing the false alarm probability. However, much of the literature in CS techniques addressing localization of sparse targets (e.g. [9], [10], [11]) fails to explain how CS techniques affect parameters of interest to radar, specifically the probabilities of false alarm and detection.

In this work, we seek to combine CS techniques with classical detection theory. CS techniques enable to solve the problem of localizing sparse targets, while classical detection theory frames radar performance in meaningful operational terms such as receiver operating characteristic (ROC) curves. In particular, we develop a new detection algorithm referred to as GLRT-Matching Pursuit (GLRT-MP). GLRT-MP combines a classical GLRT approach [1], [2], with matching pursuit [15], [14], [7]. Relying on matching pursuit concepts, GLRT-MP searches for target candidates. Detection theory is applied to test the target candidates for viability. A target that passed the detection test, is then removed from the data such that it does not interfere with subsequent searches.

This paper makes several contributions. First, we develop a GLRT for multiple targets, and provide the false alarm and detection probabilities. We then develop a CFAR detector for multiple targets and provide the false alarm and detection probabilities. Lastly, we propose the GLRT-MP, a new detection algorithm that combines concepts from matching pursuit and the GLRTs derived in this paper.

The rest of the paper is organized as follows: the signal model is found in Section II; the derivation of the GLRT when the noise power is known and unknown and the GLRT-MP algorithm are presented in Section III; numerical results are found in Section IV; finally, conclusions are drawn in Section V.

The following notation is used: boldface denotes matrices (uppercase) and vectors (lowercase); for a matrix $\mathbf{X}$; $\mathbf{X}(i,:)$ denotes the i-th row of $\mathbf{X}$, for a vector $\mathbf{x}$, $\mathbf{x}_j$ represents the j-th element of $\mathbf{x}$; $(\cdot)^T$, denotes the transpose operator; $(\cdot)^H$ denotes the complex conjugate-transpose operator; for a matrix $\mathbf{X}$, $\mathbf{P}_{\mathbf{X}}$ is the orthogonal projection matrix that projects onto the space spanned by the columns of $\mathbf{X}$; for a matrix $\mathbf{X}$, $\mathbf{P}_{\mathbf{X}}^{\perp}$ is the orthogonal projection matrix that projects onto the space orthogonal to the columns of $\mathbf{X}$; $\|\mathbf{X}\|_F$ is the Frobenius norm of the matrix $\mathbf{X}$; given a set S of indicies, $|S|$ denotes its cardinality, for a matrix $\mathbf{A}$, $\mathbf{A}_S$ is a sub-matrix of $\mathbf{A}$ that is indexed by the set S; a central chi squared random variable with a degrees of freedom (DOF) is denoted as χ_a^2 a noncentral chi squared random variable with a DOF is denoted as $\chi_a^{'2}$; a F distributed random variable with a numerator DOF and b denominator DOF is denoted as $F_{a,b}$; the noncentral F distribution with a numerator DOF and b denominator DOF and a noncentrality parameter λ is denoted as $F_{a,b}^{'}(\lambda)$; finally, for a probability distribution function X the right-tail probability at γ is denoted by $P = Q_X(\gamma)$, while $\gamma = Q_X^{-1}(P)$ denotes its inverse.

II. Signal Model

Let N sensors collect echoes from a finite train of P pulses sent by a transmitter and returned from K stationary targets. The sensors form a linear array with an aperture size of Z. The n-th receiver is placed at position z_n, measured in an arbitrary coordinate system. Targets are assumed to be in the far field, and the DOA of the k-th target is denoted θ_k.

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Following the examples in [4], [5], the DOA estimation problem may be cast in a sparse framework. Neglecting discretization errors, and assuming that all K targets comply with a grid of G DOA grid points, we define the $N \times G$ over-complete dictionary matrix $\mathbf{A} = [\mathbf{a}(\phi_1), \dots, \mathbf{a}(\phi_G)]$ where $\mathbf{a}(\phi)$ is a steering vector defined,

$$\mathbf{a}(\phi) = \left[e^{j2\pi z_1 \phi}, \dots, e^{j2\pi z_N \phi}\right]^T. \tag{1}$$

The narrowband response of the array to the p-th pulse, $\mathbf{y}(p)$, also known as a snapshot, is expressed

$$\mathbf{y}(p) = \mathbf{A}\mathbf{x}(p) + \mathbf{e}(p), \quad p = 1, 2 \dots, P \tag{2}$$

where, $\mathbf{x}(p)$ is a $G \times 1$ sparse vector of complex target gains at the p-th snapshot with $K \ll G$ nonzero entries, and $\mathbf{e}(p)$ is complex white Gaussian noise at the p-th snapshot. Entries of $\mathbf{e}(p)$ are independent and identically distributed, and each entry has zero mean and variance σ^2. We also define the measurement matrix $\mathbf{Y} = [\mathbf{y}(1), \dots \mathbf{y}(P)]$. Similarly, we define a matrix of target gains $\mathbf{X} = [\mathbf{x}(1), \dots, \mathbf{x}(P)]$ and the matrix of complex white Gaussian noise $\mathbf{E} = [\mathbf{e}(1), \dots, \mathbf{e}(P)]$. The vectors forming $\mathbf{E}$ are uncorrelated. Then, the signal model (2) may be expressed

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{E}. \tag{3}$$

Since the K targets are assumed to be stationary, the K positions of the nonzero elements of $\mathbf{x}(p)$ is the same for all $p = 1, \dots, P$ hence, $\mathbf{X}$ has K nonzero rows.

III. Detection by Compressive Sensing

The goal of detection is to determine the number of targets K, and given K, which K rows of $\mathbf{X}$ are nonzero. One approach to achieve this goal is to solve the nonconvex optimization problem

$$\min \|\mathbf{Y} - \mathbf{A}\mathbf{X}\|_F^2 + \lambda \|\mathbf{X}\|_0. \tag{4}$$

Here $\|\mathbf{X}\|_0$ is the ℓ_0 quasi-norm that counts the number of nonzero rows in $\mathbf{X}$, and λ is a regularization parameter. Numerous algorithms have been proposed in order to find an approximate solution to (4). However designing the proper value for λ often requires prior knowledge such as the number of targets K and the noise power σ^2. Even with such prior knowledge, designing λ is not an easy task.

In order to bypass designing the regularization parameter λ, we instead propose the GLRT-MP that combines concepts of the MP algorithm and detection to test candidate entries found by MP. More specifically, conventional MP iteratively identifies the positions of the nonzero rows of $\mathbf{X}$. However, without prior knowledge of K, deciding when to terminate the algorithm may be difficult. Instead, with GLRT-MP, after identifying a candidate nonzero row, GLRT-MP tests it against a threshold. The threshold is set such that false alarms do not exceed an acceptable level. If the row passes the test, it is declared to contain a target. The algorithm then searches for the next nonzero row. The algorithm terminates as soon as a selected row does not pass through the detection test.

The advantage of this approach is that it affords to trade off the regularization parameter λ, which is difficult to obtain, for a threshold parameter γ, which can be obtained from classical detection theory considerations. Since the detector plays a large role in our algorithm, we review some results from detection theory before, presenting the details of the proposed GLRT-MP algorithm.

A. GLRT - Known noise power

Here we derive the detection test for the rows of $\mathbf{X}$ when the noise power is known. Let us assume we are at iteration k, meaning that $k-1$ target have already been detected. Let S_{k-1} denote the set of indices of rows of $\mathbf{X}$ where targets have been detected. At iteration k, the GLRT-MP algorithm has $G - |S_{k-1}|$ indicies it can choose to test. It chooses the index $j = \arg\max_l \|\mathbf{a}(\phi_l)\mathbf{W}_{S_{k-1}}\|_2 / \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{a}(\phi_l)\|_2$, where $\mathbf{A}_{S_{k-1}}$ is the manifold matrix consisting of the steering vectors of the $k-1$ detected targets, $\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}$ is the projection matrix onto the space orthogonal to $\mathbf{A}_{S_{k-1}}$, and $\mathbf{W}_{S_{k-1}}$ is the orthonormal basis of $\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{Y}$. At iteration k, the GLRT-MP algorithm determines a row of $\mathbf{X}$ as a candidate target, and adds it to the set S_{k-1} to form the set S_k. Given the set S_k of previously detected targets and the current candidate target the signal model (3), is reduced to

$$\mathbf{Y} = \mathbf{A}_{S_k}\mathbf{X}_{S_k} + \mathbf{E}. \tag{5}$$

where $\mathbf{X}_{S_k}$ consists of the k rows indexed by S_k.

Next, GLRT-MP poses the following hypothesis problem

$$\begin{aligned} \mathcal{H}_0 &: \mathbf{Y} = \mathbf{A}_{S_{k-1}}\mathbf{X}_{S_{k-1}} + \mathbf{E} \\ \mathcal{H}_1 &: \mathbf{Y} = \mathbf{A}_{S_k}\mathbf{X}_{S_k} + \mathbf{E} \end{aligned} \tag{6}$$

From (5), the probability density function (pdf) of the measurement matrix $\mathbf{Y}$ conditioned on $\mathbf{X}_{S_k}$ is given by

$$p(\mathbf{Y}|\mathbf{X}_{S_k}) = \prod_{p=1}^{P} \frac{1}{\pi^N \sigma^{2N}} e^{-\frac{1}{\sigma^2}\|\mathbf{y}(p) - \mathbf{A}_{S_k}\mathbf{x}_{S_k}(p)\|_2^2} \tag{7}$$

where $\mathbf{x}_{S_k}(p)$ is a column of $\mathbf{X}_{S_k}$. The generalized likelihood ratio (GLR) is formed by taking the ratio

$$L(\mathbf{Y}, S_k) = \frac{p\left(\mathbf{Y} \middle| \widehat{\mathbf{X}}_{S_k}, \mathcal{H}_1\right)}{p\left(\mathbf{Y} \middle| \widehat{\mathbf{X}}_{S_{k-1}}, \mathcal{H}_0\right)} \tag{8}$$

where $\widehat{\mathbf{X}}_{S_k}$ is the maximum likelihood estimate (MLE) of $\mathbf{X}_{S_k}$. It can be shown by using [3], regarding $\mathbf{A}_{S_{k-1}}$ as the interference subspace and $\mathbf{A}_{S_k}$ as the signal and interference subspace that for a single snapshot $\mathbf{y}(p)$, the likelihood ratio yields

$$L(\mathbf{y}(p), S_k) = \frac{\|(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{y}(p)\|_2^2}{\sigma^2}. \tag{9}$$

For P independent snapshots, the GLRT is obtained by summing GLRTs for individual snapshots. The detection test is then,

$$\begin{aligned} L(\mathbf{Y}, S_k) &= \frac{\sum_{p=1}^{P} \|(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{y}(p)\|_2^2}{\sigma^2} \geq \gamma \\ &= \frac{\|(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{Y}\|_F^2}{\sigma^2} \geq \gamma \end{aligned} \tag{10}$$

where γ is a threshold parameter. The detector decides $\mathcal{H}_1$ if the value of $L(\mathbf{Y}, S_k)$ is greater than γ, otherwise the detector decides on $\mathcal{H}_0$.

Under $\mathcal{H}_0$, the candidate target determined by the GLRT-MP algorithm produces a likelihood ratio

$$L(\mathbf{Y}, S_k) = \max_l \; L\left(\mathbf{Y}, S_{k-1} \bigcup l\right), \tag{11}$$

where the notation $S_{k-1} \bigcup l$ is understood to mean that the set S_k is formed by adding to S_{k-1} the index of the l-th row of $\mathbf{X}$. Under $\mathcal{H}_0$, the added row does not correspond to a target, then the GLR $L(\mathbf{Y}, S_{k-1} \bigcup l)$ can be shown to be distributed according to χ^2_{2P}, the chi-squared distribution with $2P$ DOF. Since $L(\mathbf{Y}, S_k)$ is the maximum of $G - |S_{k-1}|$ random variables distributed according to χ^2_{2P}, the probability of false alarm (declaring $\mathcal{H}_1$ when $\mathcal{H}_0$ is true) is given by

$$\begin{aligned} P_F &= \Pr\left(\max_l \; L(\mathbf{Y}, S_{k-1} \bigcup l) \geq \gamma | \mathcal{H}_0\right) \\ &= \left(Q_{\chi^2_{2P}}(\gamma)\right)^{G - |S_{k-1}|} \end{aligned} \tag{12}$$

where $Q_{\chi^2_{2P}}(\cdot)$ denotes the integral over the tail of the χ^2_{2P} distribution.

Under $\mathcal{H}_1$, the GLR is distributed according to $\chi'^2_{2P}(\rho)$, the chi-squared distribution with $2P$ DOF and non-centrality parameter ρ given by

$$\rho = \frac{\|\mathbf{X}_j\|_2^2}{\sigma^2} \|(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{a}_j\|_2^2 \tag{13}$$

The probability of detection is then given by

$$\begin{aligned} P_D &= \Pr\left(L(\mathbf{Y}, S_k) \geq \gamma | \mathcal{H}_1\right) \\ &= Q_{\chi'^2_{2P}(\rho)}(\gamma). \end{aligned} \tag{14}$$

Using (12) we can obtain the threshold γ that ensures an acceptable level of false alarms α

$$\gamma = Q^{-1}_{\chi^2_{2P}}(\alpha^{1/(G - |S_{k-1}|)}). \tag{15}$$

Finally, the probability of detection for a specified α is given by

$$P_D = Q_{\chi'^2_{2P}(\rho)}(Q^{-1}_{\chi^2_{2P}}(\alpha^{1/(G - |S_{k-1}|)})). \tag{16}$$

B. GLRT - Unknown noise power

In this section, we provide a test statistic that is independent of the noise power and hence my serve as a CFAR detector. Let us assume we are at iteration k, meaning that $k-1$ targets have already been detected. At iteration k, the GLRT-MP algorithm chooses the index $j = \arg\max_l \|\mathbf{a}(\phi_l)\mathbf{W}_{S_k}\|_2 / \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{a}(\phi_l)\|_2$, as in the known noise case, and adds the index to S_{k-1} to form the set S_k. Given this set, the signal model is given by (5) and the binary hypothesis test is given by

$$\begin{aligned} \mathcal{H}_0 &: \; \mathbf{Y} = \mathbf{A}_{S_{k-1}}\mathbf{X}_{S_{k-1}} + \mathbf{E} \\ \mathcal{H}_1 &: \; \mathbf{Y} = \mathbf{A}_{S_k}\mathbf{X}_{S_k} + \mathbf{E} \end{aligned} \tag{17}$$

The GLR is given by

$$L(\mathbf{Y}, S_k) = \frac{p\left(\mathbf{Y} \middle| \widehat{\mathbf{X}}_{S_k}, \widehat{\sigma}^2_{(1)}, \mathcal{H}_1\right)}{p\left(\mathbf{Y} \middle| \widehat{\mathbf{X}}_{S_{k-1}}, \widehat{\sigma}^2_{(0)}, \mathcal{H}_0\right)} \tag{18}$$

where $\widehat{\sigma}_{(i)}$ is the MLE of the noise standard deviation σ under $\mathcal{H}_i$. It can be shown under the Gaussian assumption, [2] that (18) can be equivalently expressed as

$$L(\mathbf{Y}, S_k) = \frac{\widehat{\sigma}^2_{(0)} - \widehat{\sigma}^2_{(1)}}{\widehat{\sigma}^2_{(1)}}. \tag{19}$$

From [13] it was shown that the MLE of $\widehat{\sigma}^2_{(1)}$ assuming that $\mathbf{A}_{S_k}$ is the target manifold matrix (the set of steering vectors associated with the k targets) is given by $\widehat{\sigma}^2_{(1)} = \sum_{p=1}^{P} \mathbf{y}^H(p)(\mathbf{I} - \mathbf{P}_{\mathbf{A}_{S_k}})\mathbf{y}(p)$. Similarly, assuming that $\mathbf{A}_{S_{k-1}}$ is the target manifold matrix the $\widehat{\sigma}^2_{(0)} = \sum_{p=1}^{P} \mathbf{y}^H(p)(\mathbf{I} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{y}(p)$. Replacing the MLEs $\widehat{\sigma}^2_{(i)}$ into (19) and simplifying the expression we obtain

$$L(\mathbf{Y}, S_k) = \frac{\sum_{p=1}^{P} \mathbf{y}^H(p)(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{y}(p)}{\|\mathbf{P}^{\perp}_{\mathbf{A}_{S_k}}\mathbf{Y}\|_F^2} \geq \gamma. \tag{20}$$

Under $\mathcal{H}_1$, the numerator of (20) is distributed $\chi'^2_{2P}(\rho)$, where the noncentrality parameter can be shown to be

$$\rho = \frac{\|\mathbf{X}_j\|_2^2}{\sigma^2} \mathbf{a}_j^H(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{a}_j. \tag{21}$$

Also under $\mathcal{H}_1$ (20) is distributed $\chi'^2_{2P(N - |S_k|)}(\eta)$ where the noncentrality parameter can be shown to be

$$\eta = \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_k}}\mathbf{A}_{\overline{S}_K}\mathbf{X}_{\overline{S}_K}\|_F^2. \tag{22}$$

Therefore, under $\mathcal{H}_1$ the GLRT (20) is a ratio of two noncentral chi-squared variables, which is a doubly noncentral F random variable. Let $F^{(2)}_{2P, 2P(N - |S_{k-1}|)}(\rho, \eta)$ denote a doubly noncentral F random variable with $2P$ numerator DOF, $2P(N - |S_{k-1}|)$ denominator DOF, numerator noncentrality parameter ρ and denominator noncentrality paramater η. Then, the probability of detection is given by

$$\begin{aligned} P_D &= \Pr\left(L(\mathbf{Y}, S_k) \geq \gamma | \mathcal{H}_1\right) \\ &= Q_{F^{(2)}_{2P, 2P(N-|S_{k-1}|)}(\rho, \eta)}(\gamma). \end{aligned} \tag{23}$$

We take a moment to make several remarks about the GLRT under $\mathcal{H}_1$. First, we ask what is the probability of detection if the set S_k contains all the elements of the set $\overline{S}_K$, where $\overline{S}_K$ is the true set of K targets. Then, we can see that $\mathbf{P}^{\perp}_{\mathbf{A}_{S_k}} \mathbf{A}_{\overline{S}_K} = \mathbf{0}$, which implies that $\eta = 0$. Then the probability of detection is given by

$$P_D = Q_{F^{(2)}_{2P, 2P(N-|S_{k-1}|)}(\rho, 0)}(\gamma) = Q_{F_{2P, 2P(N-|S_{k-1}|)}(\rho)}(\gamma). \tag{24}$$

Next, we ask what is the probability of detection when S_k does not contain all the indicies in $\overline{S}_K$. Then, we see that $\mathbf{P}^{\perp}_{\mathbf{A}_{S_k}} \mathbf{A}_{\overline{S}_K} \neq \mathbf{0}$ and therefore $\eta \neq 0$, and the probability of detection is given by (23). It is argued that when $\eta \neq 0$ the probability of detection is reduced compared to when $\eta = 0$. Intuitively, when $\eta \neq 0$ the denominator of the GLRT increases reducing the value of $L(\mathbf{Y}, S_k)$ and hence reducing the probability in (24).

In words, without knowledge of the true locations of the K targets one cannot ensure that all the contributions from all K targets are removed by the orthogonal projection matrix in the denominator. If the projection matrix does not remove all the target contributions the denominator term becomes biased which introduces a noncentrality parameter η. Then, it is our goal to find a test statistic that maintains $\eta = 0$ while not requiring any knowledge of the targets.

To achieve this test statistic we take a different approach than in [2], [3]. We begin by separating the data matrix $\mathbf{Y}$ into two partitions $\mathbf{Y} = \left[\tilde{\mathbf{Y}},\ \mathbf{y}(P)\right]$ where $\tilde{\mathbf{Y}} = [\mathbf{y}(1), \mathbf{y}(2), \ldots, \mathbf{y}(P-1)]$. Using the singular value decomposition (SVD) or the eigendecomposition of $\tilde{\mathbf{Y}}$, we obtain the K dimensional signal subspace $\mathbf{U}_S$ and the $N-K$ dimensional noise subspace $\mathbf{U}_N$. We refer the reader to [12] and [4] for more information on how to obtain these subspaces.

We then approximate $\mathbf{P}^{\perp}_{\mathbf{A}_{S_k}} \approx \mathbf{P}_N$ where $\mathbf{P}_N = \mathbf{U}_N \mathbf{U}_N^H$. Notice that $\mathbf{P}_N \mathbf{A}_{\overline{S}_K} \approx \mathbf{0}$ because $\mathbf{P}_N$ projects onto the noise subspace which is orthogonal to the signal subspace where $\mathbf{A}_{\overline{S}_K}$ resides. Testing the CUT we obtain

$$L(\mathbf{y}(P), S_k) \approx \frac{\|(\mathbf{P}_{\mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{A}_{S_{k-1}}})\mathbf{y}(P)\|_2^2}{\|\mathbf{P}_N \mathbf{y}(P)\|_2^2} \geq \gamma. \tag{25}$$

Unfortunately, the numerator and denominator in (25) are no longer independent random variables in general. To remedy this, let $\mathbf{P}_S = \mathbf{U}_S \mathbf{U}_S^H$ be the projection matrix onto the signal subspace, then we make an approximation and replace $\mathbf{P}_{\mathbf{A}_{S_k}}$ with $\mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_k}} = \mathbf{P}_S \mathbf{A}_{S_k} \left(\mathbf{A}_{S_k}^H \mathbf{P}_S \mathbf{A}_{S_k}\right) \mathbf{A}_{S_k}^H \mathbf{P}_S$ which is the projection onto the subspace $\mathbf{P}_S \mathbf{A}_{S_k}$. Similarly, we replace $\mathbf{P}_{\mathbf{A}_{S_{k-1}}}$ with $\mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_{k-1}}}$. Using these replacements we obtain the approximation

$$L(\mathbf{y}(P), S_k) \approx \frac{\|(\mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_k}} - \mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_{k-1}}})\mathbf{y}(P)\|_2^2}{\|\mathbf{P}_N^{\perp} \mathbf{y}(P)\|_2^2} \geq \gamma. \tag{26}$$

It can be shown that now the numerator and denominator are independent random variables. This is because $\mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_k}} \mathbf{P}_N = \mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_{k-1}}} \mathbf{P}_N = \mathbf{0}$ due to $\mathbf{P}_S \mathbf{P}_N = \mathbf{0}$. Under $\mathcal{H}_0$, the numerator of (26) is a χ_2^2 random variable, and the denominator of (26) is a $\chi^2_{2(N-K)}$ random variable, therefore (26) is distributed as $F_{2,2(N-K)}$. Then the probability of false alarm is given by

$$\begin{aligned} P_F &= \Pr\left(L(\mathbf{y}(P), S_k) \geq \gamma | \mathcal{H}_0\right) \\ &= Q_{F_{2,2(N-K)}}(\gamma). \end{aligned} \tag{27}$$

Under $\mathcal{H}_1$, the numerator of (26) is distributed according to $\chi_2^{\prime 2}(\omega)$ with $\omega = \frac{\|\mathbf{X}_j(P)\|^2}{\sigma^2} \|(\mathbf{P}_S \mathbf{A}_{S_k} - \mathbf{P}_{\mathbf{P}_S \mathbf{A}_{S_{k-1}}})\mathbf{a}(\phi_j)\|_2^2$. The denominator of (26) is still distributed as $\chi^2_{2(N-K)}$, therefore (26) is distributed according to $F'_{2,2(N-K)}(\omega)$. The probability of detection is given by

$$\begin{aligned} P_D &= \Pr\left(L(\mathbf{y}(P), S_k) \geq \gamma | \mathcal{H}_1\right) \\ &= Q_{F'_{2,2(N-K)}(\omega)}(\gamma). \end{aligned} \tag{28}$$

Using (27), we can obtain the threshold γ that obtains a desired false alarm α as

$$\gamma = Q^{-1}_{F_{2,2(N-K)}}(\alpha). \tag{29}$$

Finally, the probability of detection for a desired false alarm α is given by

$$P_D = Q_{F'_{2,2(N-K)}(\omega)}(Q^{-1}_{F_{2,2(N-K)}}(\alpha)). \tag{30}$$

Note that our formulation requires the number of targets K to obtain the signal and noise subspaces. Empirically, we observed that overestimating the number of sources does not cause the GLRT to collapse and a slight degradation of performance was observed. Underestimating the number of sources unfortunately leads to situations where some targets maybe unrecoverable. The impact of overestimating the number of sources K will be considered in Section IV. It should also be noted that the test statistic (26) is independent of the noise power and hence is a CFAR detector.

C. GLRT-MP

In parts A and B above, we presented the GLRT for the cases of known noise and CFAR. We are now able to present the GLRT-MP algorithm. The inputs into the algorithm are the data matrix $\mathbf{Y}$, the dictionary matrix $\mathbf{A}$, and an acceptable false alarm probability α. First, we present the algorithm for known noise power σ^2.

The algorithm begins by initializing an empty set S_0, and sets the iteration counter k to one. It then searches for the index

$j = \max_l \|\mathbf{a}(\phi_l)\mathbf{W}_{S_{k-1}}\|_2 / \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{a}(\phi_l)\|_2$, where again, $\mathbf{W}_{S_{k-1}}$ is the orthonormal basis of $\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{Y}$. It then updates the set S_{k-1} to obtain the set $S_k = S_{k-1} \bigcup j$. It then calculates the appropriate threshold γ according to (15), and applies the GLRT using the set S_k. If the target passes the GLRT, then the algorithm increments k by one and the process repeats, otherwise, the set S_{k-1} is declared the set of nonzero rows in $\mathbf{X}$ and the algorithm terminates. The pseudocode for the algorithm is available below in the table Algorithm 1.

Algorithm 1 GLRT-MP - Known noise power

Input: $\mathbf{Y}$, $\mathbf{A}$, α
Initialization: Set $S_0 = \emptyset$, $k = 1$
Repeat until stopping criteria:
1. Calculate $\mathbf{W}_{S_{k-1}} = \text{Orth}(\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{Y})$
2. $j = \max_l \|\mathbf{a}(\phi_l)\mathbf{U}_{S_{k-1}} / \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{a}(\phi_l)\|_2$
3. Find suitable threshold parameter γ according to (15)
4. Calculate $L(\mathbf{Y}, S_k)$ and compare to the γ

If $L(\mathbf{Y}, S_k) < \gamma$ output S_{k-1} and terminate
Else $k = k + 1$ repeat

We now discuss the CFAR GLRT-MP when the noise power is unknown to the algorithm. The algorithm begins by initializing the empty set S_0 and sets the iteration counter k to one. Then, using the first $P - 1$ snapshots of $\mathbf{Y}$ we estimate the signal subspace $\mathbf{U}_S$ and the noise subspace $\mathbf{U}_N$. We then calculate the projection matrices $\mathbf{P}_S = \mathbf{U}_S\mathbf{U}_S^H$ and $\mathbf{P}_N = \mathbf{U}_N\mathbf{U}_N^H$. The algorithm then searches for the index $j = \max_l \|\mathbf{a}(\phi_l)\mathbf{W}_{S_{k-1}}\|_2 / \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{a}(\phi_l)\|_2$. It then calculates the appropriate threshold γ according to (29) and applies the GLRT using the set S_k. If the target passes the GLRT, then the algorithm increments k by one and the process repeats, otherwise, the set S_{k-1} is declared the set of nonzero rows in $\mathbf{X}$ and the algorithm terminates. The psuedocode for the algorithm is available below in the table Algorithm 2.

Algorithm 2 GLRT-MP - CFAR

Input: $\mathbf{Y}$, $\mathbf{A}$, α
Initialization: Set $S_0 = \emptyset$, $k = 1$
Estimate: $\mathbf{U}_S$ and $\mathbf{U}_N$ using the first $P - 1$ snapshots
Calculate: $\mathbf{P}_S = \mathbf{U}_S\mathbf{U}_S^H$ and $\mathbf{P}_N = \mathbf{U}_N\mathbf{U}_N^H$
Repeat until stopping criteria:
1. Calculate $\mathbf{W}_{S_{k-1}} = \text{Orth}(\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{Y})$
2. $j = \max_l \|\mathbf{a}(\phi_l)\mathbf{U}_{S_{k-1}} / \|\mathbf{P}^{\perp}_{\mathbf{A}_{S_{k-1}}}\mathbf{a}(\phi_l)\|_2$
3. Find suitable threshold parameter γ according to (29)
4. Calculate $L(\mathbf{Y}, S_k)$ and compare to the γ

If $L(\mathbf{Y}, S_k) < \gamma$ output S_{k-1} and terminate
Else $k = k + 1$ repeat

IV. Numerical Simulations

In this section, we present numerical results to demonstrate the potential of the GLRT-MP algorithm for target detection. Unless stated otherwise, the number of sensors is set to $N = 10$, the array aperture $Z = 10\lambda$, the number of targets is $K = 2$, the number of grid points $G = 21$, and the SNR = 12dB per snapshot, the number of snapshots $P = 1$. To produce each figure, we draw a random realization of the array sensors' positions using the uniform distribution $U \sim [0, Z]$. The sensor positions remain fixed throughout the Monte-Carlo simulations. The target positions are also randomly drawn and remain fixed throughout. The noise realizations and the phase of the targets are independently drawn from run to run.

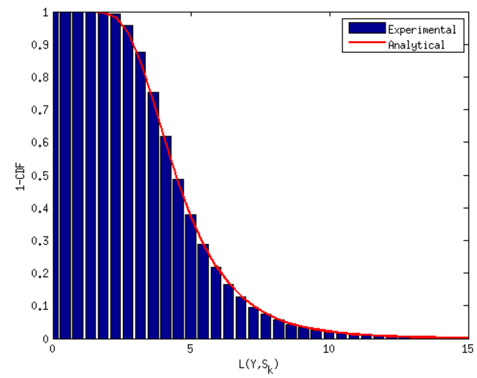

Fig. 1. Histogram of (1-CDF) of $L(\mathbf{Y}, S_k)$. Parameters used were the following, $N = 10$, $Z = 10$, SNR = 12dB, $P = 1$, $K = 2$.

In Fig. 1, we plot $(Q_{\chi^2_{2P}}(\gamma))^{(G-|S_{k-1}|)}$ and the experimental complementary function (1-CDF) of $L(\mathbf{Y}, S_k)$ to examine how well our approximation in (12) fares to simulation. It can be seen from the figure, that the analytical expression $(Q_{\chi^2_{2P}}(\gamma))^{(G-|S_{k-1}|)}$ closely resembles the simulation. This suggests that designing a threshold parameter from the analytical expressions provided in this paper results in a probability of false alarm that is very close to the desired false alarm probability.

In Fig. 2, we plot a ROC curve for both the conventional beamformer and the GLRT-MP algorithm for $P = 1$, when the noise power is known. It is seen that the ROC curve of the GLRT-MP is to the left of the beamformer showing that the GLRT-MP can achieve the same probability of detection for a much lower probability of false alarm.

In Fig. 3, we plot the ROC for the GLRT-MP algorithm for $P = 25$, in the CFAR case for various estimations of the dimensionality of the signal subspace when the true dimensionality of the signal subspace is $K = 2$. We denote $\hat{K}$ as the estimate of the dimensionality K. When $\hat{K} = 2$ the GLRT-MP terminates once two targets have been estimated because $\hat{K}$ provides an upperbound on the number of targets. Therefore no false alarms occur as long as the algorithm correctly estimates the targets. We observe that $\hat{K} = 3$ achieves the best performance of the three curves. It is also observed as the estimate of the dimensionality is farther from $\hat{K} = 3$ the curves move slightly to the right suggesting a small increase in false alarm probability.

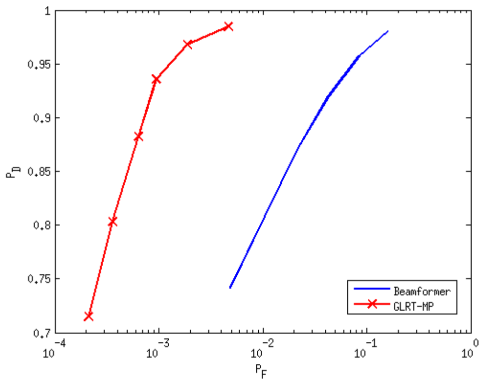

Fig. 2. ROC curves of the GLRT-MP and the conventional beamformer. Parameters used were the following, $N = 10$, $Z = 10$, SNR = 12dB, $P = 1$, $K = 2$.

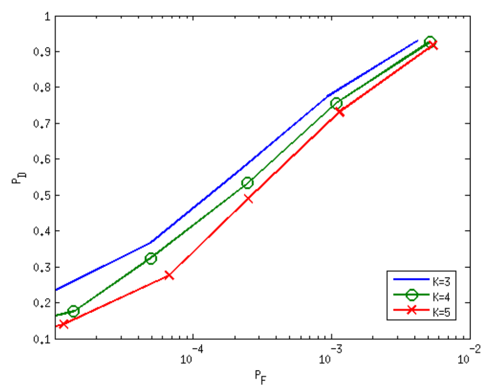

Fig. 3. ROC curves for various estimates of the dimensionality of the signal subspace. Parameters used were the following, $N = 10$, $Z = 10$, SNR = 12dB, $P = 1$, $K = 2$.

V. Conclusions

In this paper, we developed a GLRT for multiple targets when the noise power is known and provided the false alarm and detection probabilities. We then build upon the known noise case and develop a CFAR detector for multiple targets and provide the false alarm and detection probabilities. We then propose the GLRT-MP algorithm, a detector that combines matching pursuit concepts and the GLRT derived in this paper. Numerical results show that the analytical expressions derived in this paper closely match the simulation results and outperforms the conventional beamformer.

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